EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

SOLUTION The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

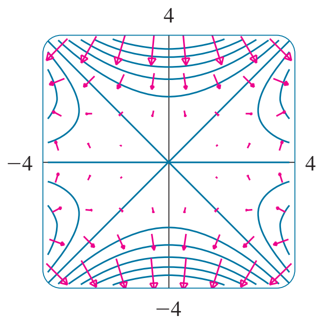


FIGURE 15

Figure 15 shows a contour map of f with the gradient vector field. Notice that the gradient vectors are perpendicular to the level curves, as we would expect from Section 11.6. Notice also that the gradient vectors are long where the level curves are close to each other and short where they are farther apart. That's because the length of the gradient vector is the value of the directional derivative of f and close level curves indicate a steep graph. ■

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field $\mathbf{F}$ in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}\\ &= \mathbf{F}(x, y, z)\end{aligned}$$

In Sections 13.3 and 13.5 we will learn how to tell whether or not a given vector field is conservative.

13.1 Exercises

1–10 ■ Sketch the vector field $\mathbf{F}$ by drawing a diagram like Figure 5 or Figure 9.

1. $\mathbf{F}(x, y) = \frac{1}{2}(\mathbf{i} + \mathbf{j})$

2. $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$

3. $\mathbf{F}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}$

4. $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$

5. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

6. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

7. $\mathbf{F}(x, y, z) = \mathbf{j}$

8. $\mathbf{F}(x, y, z) = z\,\mathbf{j}$

9. $\mathbf{F}(x, y, z) = y\,\mathbf{j}$

10. $\mathbf{F}(x, y, z) = \mathbf{j} - \mathbf{i}$

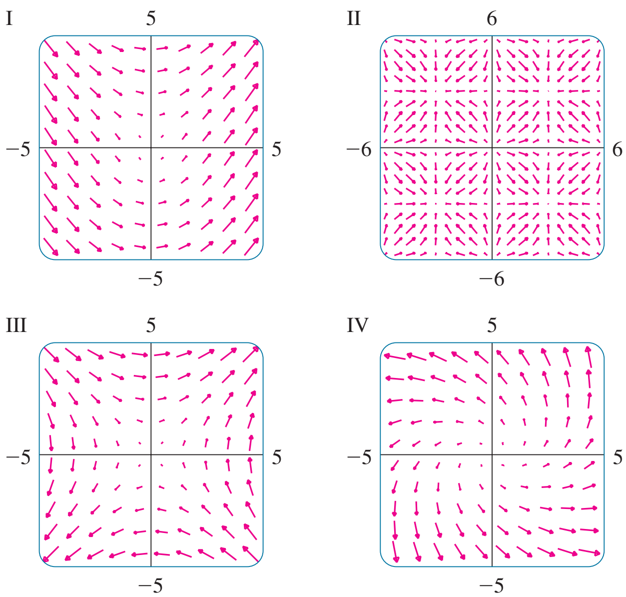
11–14 ■ Match the vector fields $\mathbf{F}$ with the plots labeled I–IV. Give reasons for your choices.

11. $\mathbf{F}(x, y) = \langle y, x\rangle$

12. $\mathbf{F}(x, y) = \langle 2x - 3y, 2x + 3y\rangle$

13. $\mathbf{F}(x, y) = \langle \sin x, \sin y\rangle$

14. $\mathbf{F}(x, y) = \langle \ln(1 + x^2 + y^2), x\rangle$



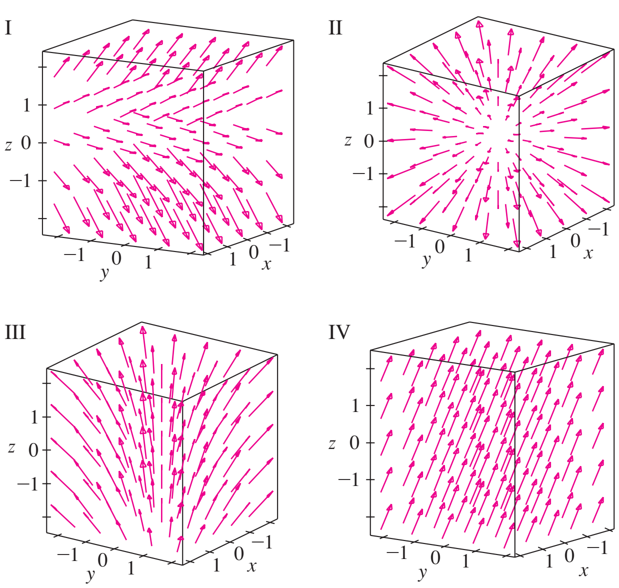
15–18 ■ Match the vector fields $\mathbf{F}$ on $\mathbb{R}^3$ with the plots labeled I–IV. Give reasons for your choices.

15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$

17. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$

18. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$



CAS **19.** If you have a CAS that plots vector fields (the command is fieldplot in Maple and PlotVectorField in Mathematica), use it to plot

$$\mathbf{F}(x, y) = (y^2 - 2xy)\,\mathbf{i} + (3xy - 6x^2)\,\mathbf{j}$$

Explain the appearance by finding the set of points (x, y) such that $\mathbf{F}(x, y) = \mathbf{0}$.

CAS **20.** Let $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$, where $\mathbf{x} = \langle x, y \rangle$ and $r = |\mathbf{x}|$. Use a CAS to plot this vector field in various domains until you can see what is happening. Describe the appearance of the plot and explain it by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

21–24 ■ Find the gradient vector field of f.

21. $f(x, y) = \ln(x + 2y)$

22. $f(x, y) = x^{\alpha} e^{-\beta x}$

23. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

24. $f(x, y, z) = x \cos(y/z)$

25–26 ■ Find the gradient vector field ∇f of f and sketch it.

25. $f(x, y) = xy - 2x$

26. $f(x, y) = \frac{1}{4}(x + y)^2$

CAS **27–28** ■ Plot the gradient vector field of f together with a contour map of f. Explain how they are related to each other.

27. $f(x, y) = \sin x + \sin y$

28. $f(x, y) = \sin(x + y)$

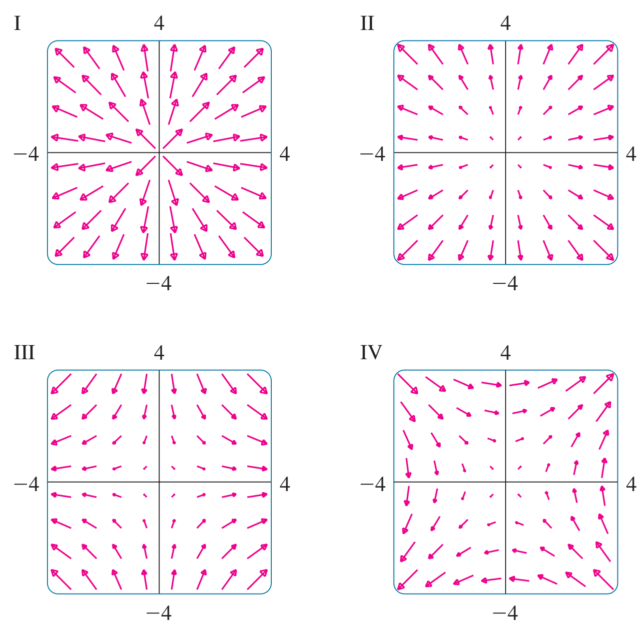
29–32 ■ Match the functions f with the plots of their gradient vector fields (labeled I–IV). Give reasons for your choices.

29. $f(x, y) = xy$

30. $f(x, y) = x^2 - y^2$

31. $f(x, y) = x^2 + y^2$

32. $f(x, y) = \sqrt{x^2 + y^2}$



33. The **flow lines** (or **streamlines**) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus, the vectors in a vector field are tangent to the flow lines.

(a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?

(b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $dx/dt = x$ and $dy/dt = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point (1, 1).

34. (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$ and then sketch some flow lines. What shape do these flow lines appear to have?

(b) If parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? Deduce that $dy/dx = x$.

(c) If a particle starts at the origin in the velocity field given by $\mathbf{F}$, find an equation of the path it follows.

13.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C. Such integrals are called *line integrals,* although "curve integrals" would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

We start with a plane curve C given by the parametric equations

1 $$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. See Section 10.2.] If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$. (See Figure 1.) We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.) Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

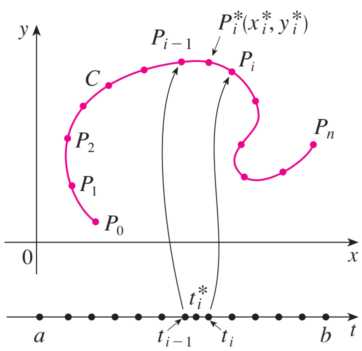


FIGURE 1

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

if this limit exists.

In Section 6.3 we found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

▲ Figure 13 shows the twisted cubic C in Example 8 and some typical vectors acting at three points on C.

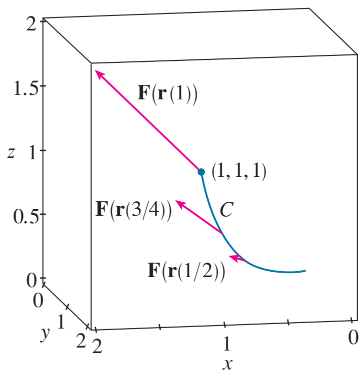


FIGURE 13

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

SOLUTION We have

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\,\mathbf{i} + t^5\,\mathbf{j} + t^4\,\mathbf{k}$$

Thus
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^1 (t^3 + 5t^6)\,dt = \frac{t^4}{4} + \frac{5t^7}{7}\bigg]_0^1 = \frac{27}{28}$$

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. We use Definition 13 to compute its line integral along C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k})\,dt$$

$$= \int_a^b [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)]\,dt$$

But this last integral is precisely the line integral in (10). Therefore, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz \qquad \text{where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

For example, the integral $\int_C y\,dx + z\,dy + x\,dz$ in Example 6 could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$

13.2 Exercises

1–12 ■ Evaluate the line integral, where C is the given curve.

1. $\int_C y\,ds$, $\quad C: x = t^2,\ y = t,\ 0 \leqslant t \leqslant 2$

2. $\int_C (y/x)\,ds$, $\quad C: x = t^4,\ y = t^3,\ \frac{1}{2} \leqslant t \leqslant 1$

3. $\int_C xy^4\,ds$, $\quad C$ is the right half of the circle $x^2 + y^2 = 16$

4. $\int_C \sin x\,dx$,
C is the arc of the curve $x = y^4$ from $(1, -1)$ to $(1, 1)$

5. $\int_C xy\,dx + (x - y)\,dy$, $\quad C$ consists of line segments from $(0, 0)$ to $(2, 0)$ and from $(2, 0)$ to $(3, 2)$

6. $\int_C x\sqrt{y}\, dx + 2y\sqrt{x}\, dy$,
C consists of the shortest arc of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$ and the line segment from $(0, 1)$ to $(4, 3)$

7. $\int_C xy^3\, ds$,
C: $x = 4 \sin t,\ y = 4 \cos t,\ z = 3t,\ 0 \leqslant t \leqslant \pi/2$

8. $\int_C x^2 z\, ds$, C is the line segment from $(0, 6, -1)$ to $(4, 1, 5)$

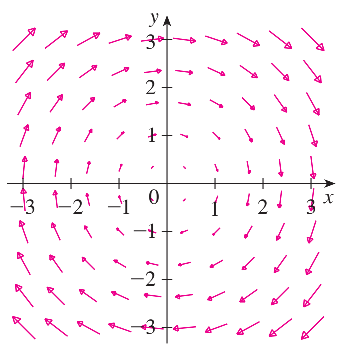
9. $\int_C xe^{yz}\, ds$, C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$

10. $\int_C yz\, dy + xy\, dz$, C: $x = \sqrt{t},\ y = t,\ z = t^2,\ 0 \leqslant t \leqslant 1$

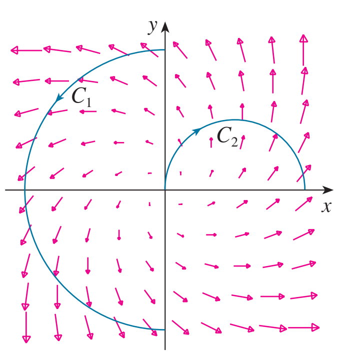
11. $\int_C z^2\, dx - z\, dy + 2y\, dz$,
C consists of line segments from $(0, 0, 0)$ to $(0, 1, 1)$, from $(0, 1, 1)$ to $(1, 2, 3)$, and from $(1, 2, 3)$ to $(1, 2, 4)$

12. $\int_C yz\, dx + xz\, dy + xy\, dz$,
C consists of line segments from $(0, 0, 0)$ to $(2, 0, 0)$, from $(2, 0, 0)$ to $(1, 3, -1)$, and from $(1, 3, -1)$ to $(1, 3, 0)$

13. Let **F** be the vector field shown in the figure.
(a) If C_1 is the vertical line segment from $(-3, -3)$ to $(-3, 3)$, determine whether $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.
(b) If C_2 is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.



14. The figure shows a vector field **F** and two curves C_1 and C_2. Are the line integrals of **F** over C_1 and C_2 positive, negative, or zero? Explain.



15–18 ■ Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

15. $\mathbf{F}(x, y) = x^2 y^3\, \mathbf{i} - y\sqrt{x}\, \mathbf{j}$,
$\mathbf{r}(t) = t^2\, \mathbf{i} - t^3\, \mathbf{j}$, $0 \leqslant t \leqslant 1$

16. $\mathbf{F}(x, y, z) = yz\, \mathbf{i} + xz\, \mathbf{j} + xy\, \mathbf{k}$,
$\mathbf{r}(t) = t\, \mathbf{i} + t^2\, \mathbf{j} + t^3\, \mathbf{k}$, $0 \leqslant t \leqslant 2$

17. $\mathbf{F}(x, y, z) = \sin x\, \mathbf{i} + \cos y\, \mathbf{j} + xz\, \mathbf{k}$,
$\mathbf{r}(t) = t^3\, \mathbf{i} - t^2\, \mathbf{j} + t\, \mathbf{k}$, $0 \leqslant t \leqslant 1$

18. $\mathbf{F}(x, y, z) = x^2\, \mathbf{i} + xy\, \mathbf{j} + z^2\, \mathbf{k}$,
$\mathbf{r}(t) = \sin t\, \mathbf{i} + \cos t\, \mathbf{j} + t^2\, \mathbf{k}$, $0 \leqslant t \leqslant \pi/2$

CAS **19–20** ■ Use a graph of the vector field **F** and the curve C to guess whether the line integral of **F** over C is positive, negative, or zero. Then evaluate the line integral.

19. $\mathbf{F}(x, y) = (x - y)\, \mathbf{i} + xy\, \mathbf{j}$,
C is the arc of the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$ to $(0, -2)$

20. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\, \mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\, \mathbf{j}$,
C is the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$

21. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = e^{x-1}\, \mathbf{i} + xy\, \mathbf{j}$ and C is given by $\mathbf{r}(t) = t^2\, \mathbf{i} + t^3\, \mathbf{j}$, $0 \leqslant t \leqslant 1$.
(b) Illustrate part (a) by using a graphing calculator or computer to graph C and the vectors from the vector field corresponding to $t = 0$, $1/\sqrt{2}$, and 1 (as in Figure 13).

22. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x\, \mathbf{i} - z\, \mathbf{j} + y\, \mathbf{k}$ and C is given by $\mathbf{r}(t) = 2t\, \mathbf{i} + 3t\, \mathbf{j} - t^2\, \mathbf{k}$, $-1 \leqslant t \leqslant 1$.
(b) Illustrate part (a) by using a computer to graph C and the vectors from the vector field corresponding to $t = \pm 1$ and $\pm \frac{1}{2}$ (as in Figure 13).

CAS **23.** Find the exact value of $\int_C x^3 y^5\, ds$, where C is the part of the astroid $x = \cos^3 t$, $y = \sin^3 t$ in the first quadrant.

24. (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\, \mathbf{i} + xy\, \mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction.
CAS (b) Use a computer algebra system to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

25. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x \geqslant 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

26. Find the mass and center of mass of a thin wire in the shape of a quarter-circle $x^2 + y^2 = r^2$, $x \geqslant 0$, $y \geqslant 0$, if the density function is $\rho(x, y) = x + y$.

27. (a) Write the formulas similar to Equations 4 for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire with density function $\rho(x, y, z)$ in the shape of a space curve C.
(b) Find the center of mass of a wire in the shape of the helix $x = 2 \sin t$, $y = 2 \cos t$, $z = 3t$, $0 \leqslant t \leqslant 2\pi$, if the density is a constant k.

28. Find the mass and center of mass of a wire in the shape of the helix $x = t$, $y = \cos t$, $z = \sin t$, $0 \leqslant t \leqslant 2\pi$, if the density at any point is equal to the square of the distance from the origin.

29. If a wire with linear density $\rho(x, y)$ lies along a plane curve C, its **moments of inertia** about the x- and y-axes are defined as

$$I_x = \int_C y^2\rho(x, y)\, ds \qquad I_y = \int_C x^2\rho(x, y)\, ds$$

Find the moments of inertia for the wire in Example 3.

30. If a wire with linear density $\rho(x, y, z)$ lies along a space curve C, its **moments of inertia** about the x-, y-, and z-axes are defined as

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds$$

Find the moments of inertia for the wire in Exercise 27.

31. Find the work done by the force field $\mathbf{F}(x, y) = x\,\mathbf{i} + (y + 2)\,\mathbf{j}$ in moving an object along an arch of the cycloid $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$.

32. Find the work done by the force field $\mathbf{F}(x, y) = x \sin y\,\mathbf{i} + y\,\mathbf{j}$ on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

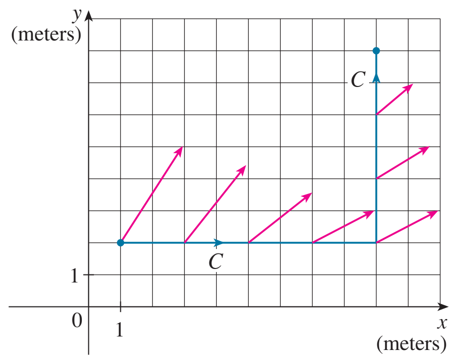
33. Find the work done by the force field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yx\,\mathbf{j} + zy\,\mathbf{k}$ on a particle that moves along the curve $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j} + t^4\,\mathbf{k}$, $0 \leqslant t \leqslant 1$.

34. The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. (See Example 5 in Section 13.1.) Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

35. A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions, how much work is done by the man against gravity in climbing to the top?

36. Suppose there is a hole in the can of paint in Exercise 35 and 9 lb of paint leak steadily out of the can during the man's ascent. How much work is done?

37. An object moves along the curve C shown in the figure from $(1, 2)$ to $(9, 8)$. The lengths of the vectors in the force field $\mathbf{F}$ are measured in newtons by the scales on the axes. Estimate the work done by $\mathbf{F}$ on the object.

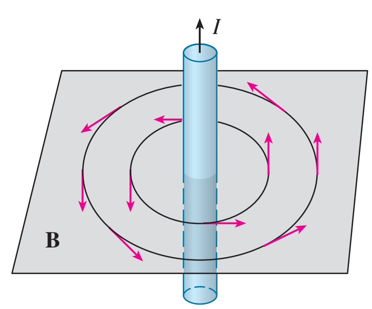


38. Experiments show that a steady current I in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where I is the net current that passes through any surface bounded by a closed curve C and μ_0 is a constant called the permeability of free space. By taking C to be a circle with radius r, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance r from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$



Therefore

15 $$W = \tfrac{1}{2}m\,|\mathbf{v}(b)|^2 - \tfrac{1}{2}m\,|\mathbf{v}(a)|^2$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity.

The quantity $\frac{1}{2}m\,|\mathbf{v}(t)|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore, we can rewrite Equation 15 as

16 $$W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that $\mathbf{F}$ is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$. Then by Theorem 2 we have

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} \\ &= -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] \\ &= P(A) - P(B) \end{aligned}$$

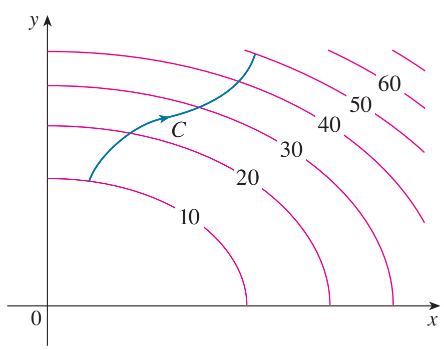
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

13.3 Exercises

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. A table of values of a function f with continuous gradient is given. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C has parametric equations $x = t^2 + 1$, $y = t^3 + t$, $0 \leqslant t \leqslant 1$.

x \ y	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

3–10 ■ Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

3. $\mathbf{F}(x, y) = (6x + 5y)\,\mathbf{i} + (5x + 4y)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = (x^3 + 4xy)\,\mathbf{i} + (4xy - y^3)\,\mathbf{j}$

5. $\mathbf{F}(x, y) = xe^y\,\mathbf{i} + ye^x\,\mathbf{j}$

6. $\mathbf{F}(x, y) = e^y\,\mathbf{i} + xe^y\,\mathbf{j}$

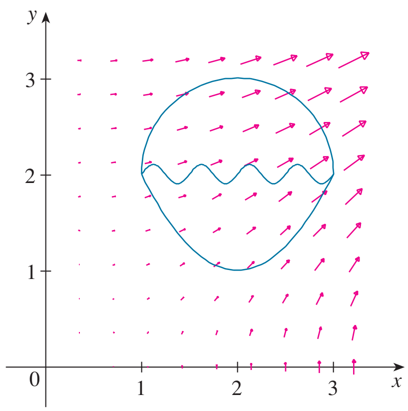
7. $\mathbf{F}(x, y) = (2x\cos y - y\cos x)\,\mathbf{i} + (-x^2\sin y - \sin x)\,\mathbf{j}$

8. $\mathbf{F}(x, y) = (1 + 2xy + \ln x)\,\mathbf{i} + x^2\,\mathbf{j}$

9. $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$

10. $\mathbf{F}(x, y) = (ye^{xy} + 4x^3y)\,\mathbf{i} + (xe^{xy} + x^4)\,\mathbf{j}$

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at (1, 2) and end at (3, 2).
(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.
(b) What is this common value?



12–18 ■ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

12. $\mathbf{F}(x, y) = y\,\mathbf{i} + (x + 2y)\,\mathbf{j}$,
C is the upper semicircle that starts at (0, 1) and ends at (2, 1)

13. $\mathbf{F}(x, y) = x^3y^4\,\mathbf{i} + x^4y^3\,\mathbf{j}$,
C: $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 + t^3)\,\mathbf{j}, \quad 0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y) = e^{2y}\,\mathbf{i} + (1 + 2xe^{2y})\,\mathbf{j}$,
C: $\mathbf{r}(t) = te^t\,\mathbf{i} + (1 + t)\,\mathbf{j}, \quad 0 \leqslant t \leqslant 1$

15. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

16. $\mathbf{F}(x, y, z) = (2xz + y^2)\,\mathbf{i} + 2xy\,\mathbf{j} + (x^2 + 3z^2)\,\mathbf{k}$,
C: $x = t^2, \; y = t + 1, \; z = 2t - 1, \quad 0 \leqslant t \leqslant 1$

17. $\mathbf{F}(x, y, z) = y^2 \cos z\,\mathbf{i} + 2xy \cos z\,\mathbf{j} - xy^2 \sin z\,\mathbf{k}$,
C: $\mathbf{r}(t) = t^2\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \pi$

18. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + xe^y\,\mathbf{j} + (z + 1)e^z\,\mathbf{k}$,
C: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

19–20 ■ Show that the line integral is independent of path and evaluate the integral.

19. $\int_C 2x \sin y\, dx + (x^2 \cos y - 3y^2)\, dy$,
C is any path from $(-1, 0)$ to $(5, 1)$

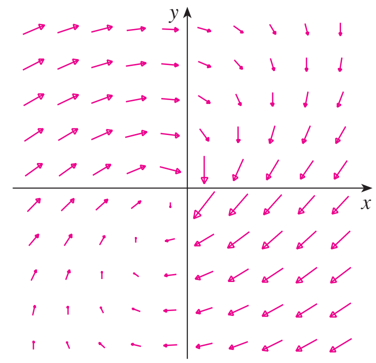
20. $\int_C (2y^2 - 12x^3y^3)\, dx + (4xy - 9x^4y^2)\, dy$,
C is any path from (1, 1) to (3, 2)

21–22 ■ Find the work done by the force field $\mathbf{F}$ in moving an object from P to Q.

21. $\mathbf{F}(x, y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}; \quad P(0, 0), \; Q(2, 1)$

22. $\mathbf{F}(x, y) = (y^2/x^2)\,\mathbf{i} - (2y/x)\,\mathbf{j}; \quad P(1, 1), \; Q(4, -2)$

23. Is the vector field shown in the figure conservative? Explain.



CAS **24–25** ■ From a plot of $\mathbf{F}$ guess whether it is conservative. Then determine whether your guess is correct.

24. $\mathbf{F}(x, y) = (2xy + \sin y)\,\mathbf{i} + (x^2 + x \cos y)\,\mathbf{j}$

25. $\mathbf{F}(x, y) = \dfrac{(x - 2y)\,\mathbf{i} + (x - 2)\,\mathbf{j}}{\sqrt{1 + x^2 + y^2}}$

26. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ (b) $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

27. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

28. Use Exercise 27 to show that the line integral $\int_C y\, dx + x\, dy + xyz\, dz$ is not independent of path.

29–32 ■ Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

29. $\{(x, y) \mid x > 0, y > 0\}$

30. $\{(x, y) \mid x \neq 0\}$

31. $\{(x, y) \mid 1 < x^2 + y^2 < 4\}$

32. $\{(x, y) \mid x^2 + y^2 \leqslant 1 \text{ or } 4 \leqslant x^2 + y^2 \leqslant 9\}$

33. Let $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.
(a) Show that $\partial P/\partial y = \partial Q/\partial x$.
(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. [*Hint:* Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from (1, 0) to $(-1, 0)$.] Does this contradict Theorem 6?

34. (a) Suppose that $\mathbf{F}$ is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Find the work done by $\mathbf{F}$ in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 4 in Section 13.1. Use part (a) to find the work done by the gravitational field when Earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the Sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}$ N·m^2/kg^2.)

(c) Another example of an inverse square field is the electric field $\mathbf{E} = \varepsilon qQ\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 5 in Section 13.1. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric field. (Use the value $\varepsilon = 8.985 \times 10^{10}$.)

13.4 Green's Theorem

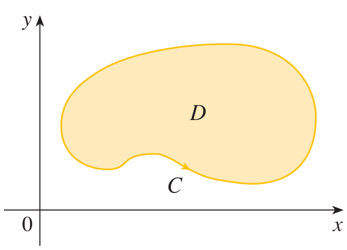


FIGURE 1

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 1. We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus, if C is given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. (See Figure 2.)

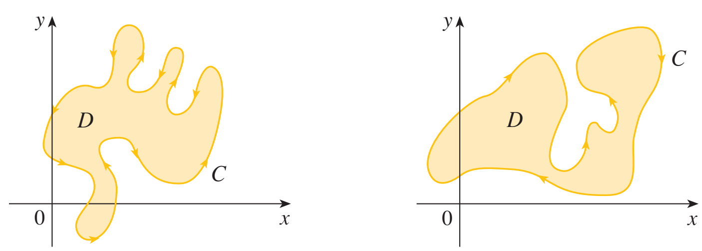


FIGURE 2 (a) Positive orientation (b) Negative orientation

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

NOTE • The notation

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \oint_C P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary

13.4 Exercises

1–4 ■ Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C xy^2\,dx + x^3\,dy$,
C is the rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 3)$, and $(0, 3)$

2. $\oint_C y\,dx - x\,dy$,
C is the circle with center the origin and radius 1

3. $\oint_C xy\,dx + x^2y^3\,dy$,
C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$

4. $\oint_C (x^2 + y^2)\,dx + 2xy\,dy$, C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$ and the line segments from $(2, 4)$ to $(0, 4)$ and from $(0, 4)$ to $(0, 0)$

CAS **5–6** ■ Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

5. $P(x, y) = x^4y^5$, $Q(x, y) = -x^7y^6$,
C is the circle $x^2 + y^2 = 1$

6. $P(x, y) = y^2 \sin x$, $Q(x, y) = x^2 \sin y$,
C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the line segment from $(1, 1)$ to $(0, 0)$

7–16 ■ Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

7. $\int_C e^y\,dx + 2xe^y\,dy$,
C is the square with sides $x = 0$, $x = 1$, $y = 0$, and $y = 1$

8. $\int_C x^2y^2\,dx + 4xy^3\,dy$,
C is the triangle with vertices $(0, 0)$, $(1, 3)$, and $(0, 3)$

9. $\int_C \left(y + e^{\sqrt{x}}\right)dx + (2x + \cos y^2)\,dy$,
C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

10. $\int_C (y^2 - \tan^{-1}x)\,dx + (3x + \sin y)\,dy$,
C is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$

11. $\int_C y^3\,dx - x^3\,dy$, C is the circle $x^2 + y^2 = 4$

12. $\int_C \sin y\,dx + x\cos y\,dy$, C is the ellipse $x^2 + xy + y^2 = 1$

13. $\int_C xy\,dx + 2x^2\,dy$,
C consists of the line segment from $(-2, 0)$ to $(2, 0)$ and the top half of the circle $x^2 + y^2 = 4$

14. $\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy$,
C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$

15. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (y^2 - x^2y)\,\mathbf{i} + xy^2\,\mathbf{j}$,
C consists of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $\left(\sqrt{2}, \sqrt{2}\right)$ and the line segments from $\left(\sqrt{2}, \sqrt{2}\right)$ to $(0, 0)$ and from $(0, 0)$ to $(2, 0)$

16. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = y^6\,\mathbf{i} + xy^5\,\mathbf{j}$,
C is the ellipse $4x^2 + y^2 = 1$

17. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

18. A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$.

19–20 ■ Find the area of the given region using one of the formulas in Equations 5.

19. The region bounded by the hypocycloid with vector equation $\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$

20. The region bounded by the curve with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$

21. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$$

(c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

22. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \qquad \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx$$

where A is the area of D.

23. Use Exercise 22 to find the centroid of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

24. Use Exercise 22 to find the centroid of a semicircular region of radius a.

25. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3}\oint_C y^3\,dx \qquad I_y = \frac{\rho}{3}\oint_C x^3\,dy$$

26. Use Exercise 25 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 4 in Section 12.5.)

27. If $\mathbf{F}$ is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

28. Complete the proof of the special case of Green's Theorem by proving Equation 3.

29. Use Green's Theorem to prove the change of variables formula for a double integral (Formula 12.9.9) for the case where $f(x, y) = 1$:

$$\iint_R dx\, dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Here R is the region in the xy-plane that corresponds to the region S in the uv-plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

[*Hint:* Note that the left side is $A(R)$ and apply the first part of Equation 5. Convert the line integral over ∂R to a line integral over ∂S and apply Green's Theorem in the uv-plane.]

13.5 Curl and Divergence

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.

Curl

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{1} \quad \text{curl}\,\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

As an aid to our memory, let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\ &= \text{curl}\,\mathbf{F} \end{aligned}$$

by Green's Theorem. But the integrand in this double integral is just the divergence of **F**. So we have a second vector form of Green's Theorem.

13
$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

This version says that the line integral of the normal component of **F** along C is equal to the double integral of the divergence of **F** over the region D enclosed by C.

13.5 Exercises

1–6 ■ Find (a) the curl and (b) the divergence of the vector field.

1. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$

2. $\mathbf{F}(x, y, z) = (x - 2z)\,\mathbf{i} + (x + y + z)\,\mathbf{j} + (x - 2y)\,\mathbf{k}$

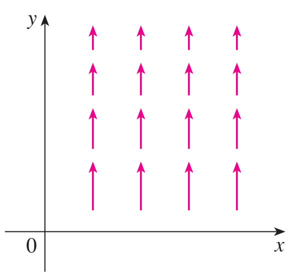
3. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} - x^2y\,\mathbf{k}$

4. $\mathbf{F}(x, y, z) = xe^y\,\mathbf{j} + ye^z\,\mathbf{k}$

5. $\mathbf{F}(x, y, z) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + z\,\mathbf{k}$

6. $\mathbf{F}(x, y, z) = \dfrac{x}{x^2 + y^2 + z^2}\,\mathbf{i} + \dfrac{y}{x^2 + y^2 + z^2}\,\mathbf{j} + \dfrac{z}{x^2 + y^2 + z^2}\,\mathbf{k}$

7–9 ■ The vector field **F** is shown in the xy-plane and looks the same in all other horizontal planes. (In other words, **F** is independent of z and its z-component is 0.)
(a) Is div **F** positive, negative, or zero? Explain.
(b) Determine whether curl $\mathbf{F} = \mathbf{0}$. If not, in which direction does curl **F** point?

7.


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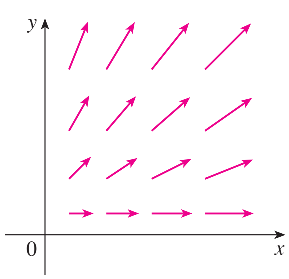

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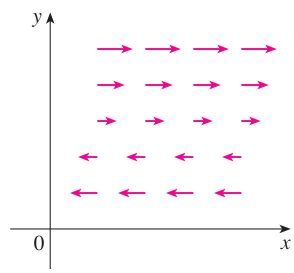


10. Let f be a scalar field and **F** a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.
(a) curl f (b) grad f
(c) div **F** (d) curl(grad f)
(e) grad **F** (f) grad(div **F**)
(g) div(grad f) (h) grad(div f)
(i) curl(curl **F**) (j) div(div **F**)
(k) (grad f) × (div **F**) (l) div(curl(grad f))

11–16 ■ Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

11. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$

12. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

13. $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + y^2\,\mathbf{k}$

14. $\mathbf{F}(x, y, z) = xy^2z^3\,\mathbf{i} + 2x^2yz^3\,\mathbf{j} + 3x^2y^2z^2\,\mathbf{k}$

15. $\mathbf{F}(x, y, z) = e^x\,\mathbf{i} + e^z\,\mathbf{j} + e^y\,\mathbf{k}$

16. $\mathbf{F}(x, y, z) = yze^{xz}\,\mathbf{i} + e^{xz}\,\mathbf{j} + xye^{xz}\,\mathbf{k}$

17. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = xy^2\,\mathbf{i} + yz^2\,\mathbf{j} + zx^2\,\mathbf{k}$? Explain.

18. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = yz\,\mathbf{i} + xyz\,\mathbf{j} + xy\,\mathbf{k}$? Explain.

19. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$$

where f, g, h are differentiable functions, is irrotational.

20. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$$

is incompressible.

21–27 ■ Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$, $\mathbf{F}\cdot\mathbf{G}$, and $\mathbf{F}\times\mathbf{G}$ are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\mathbf{F}(x, y, z)$$
$$(\mathbf{F}\cdot\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\cdot\mathbf{G}(x, y, z)$$
$$(\mathbf{F}\times\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\times\mathbf{G}(x, y, z)$$

21. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

22. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$

23. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$

24. $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$

25. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$

26. $\operatorname{div}(\nabla f\times\nabla g) = 0$

27. $\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$

28–30 ■ Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$.

28. Verify each identity.
(a) $\nabla\cdot\mathbf{r} = 3$ (b) $\nabla\cdot(r\mathbf{r}) = 4r$
(c) $\nabla^2 r^3 = 12r$

29. Verify each identity.
(a) $\nabla r = \mathbf{r}/r$ (b) $\nabla\times\mathbf{r} = \mathbf{0}$
(c) $\nabla(1/r) = -\mathbf{r}/r^3$ (d) $\nabla\ln r = \mathbf{r}/r^2$

30. If $\mathbf{F} = \mathbf{r}/r^p$, find $\operatorname{div}\mathbf{F}$. Is there a value of p for which $\operatorname{div}\mathbf{F} = 0$?

31. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$$

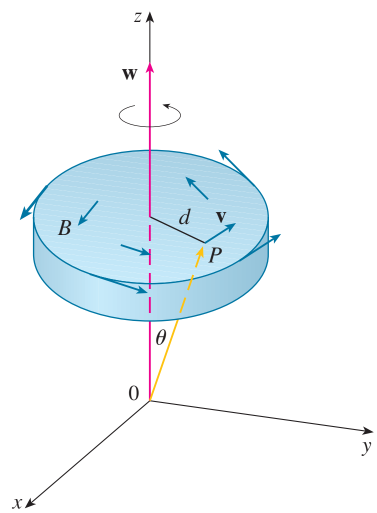
where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g\cdot\mathbf{n} = D_{\mathbf{n}}\,g$ occurs in the line integral. This is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g.)

32. Use Green's first identity (Exercise 31) to prove **Green's second identity**:

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\,dA = \oint_C (f\nabla g - g\nabla f)\cdot\mathbf{n}\,ds$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

33. This exercise demonstrates a connection between the curl vector and rotations. Let B be a rigid body rotating about the z-axis. The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of B, that is, the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} = \langle x, y, z\rangle$ be the position vector of P.
(a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.
(b) Show that $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$.
(c) Show that $\operatorname{curl}\mathbf{v} = 2\mathbf{w}$.



34. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div}\mathbf{E} = 0 \qquad \operatorname{div}\mathbf{H} = 0$$
$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \qquad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla\times(\nabla\times\mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2\mathbf{E}}{\partial t^2}$

(b) $\nabla\times(\nabla\times\mathbf{H}) = -\dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$

(c) $\nabla^2\mathbf{E} = \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ [*Hint:* Use Exercise 27.]

(d) $\nabla^2\mathbf{H} = \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$

13.6 Exercises

1. Let S be the cube with vertices $(\pm1, \pm1, \pm1)$. Approximate $\iint_S \sqrt{x^2 + 2y^2 + 3z^2}\, dS$ by using a Riemann sum as in Definition 1, taking the patches S_{ij} to be the squares that are the faces of the cube and the points P_{ij}^* to be the centers of the squares.

2. A surface S consists of the cylinder $x^2 + y^2 = 1$, $-1 \leqslant z \leqslant 1$, together with its top and bottom disks. Suppose you know that f is a continuous function with $f(\pm1, 0, 0) = 2$, $f(0, \pm1, 0) = 3$, and $f(0, 0, \pm1) = 4$. Estimate the value of $\iint_S f(x, y, z)\, dS$ by using a Riemann sum, taking the patches S_{ij} to be four quarter-cylinders and the top and bottom disks.

3. Let H be the hemisphere $x^2 + y^2 + z^2 = 50$, $z \geqslant 0$, and suppose f is a continuous function with $f(3, 4, 5) = 7$, $f(3, -4, 5) = 8$, $f(-3, 4, 5) = 9$, and $f(-3, -4, 5) = 12$. By dividing H into four patches, estimate the value of $\iint_H f(x, y, z)\, dS$.

4. Suppose that $f(x, y, z) = g\left(\sqrt{x^2 + y^2 + z^2}\right)$, where g is a function of one variable such that $g(2) = -5$. Evaluate $\iint_S f(x, y, z)\, dS$, where S is the sphere $x^2 + y^2 + z^2 = 4$.

5–18 ■ Evaluate the surface integral.

5. $\iint_S yz\, dS$,
S is the surface with parametric equations $x = uv$, $y = u + v$, $z = u - v$, $u^2 + v^2 \leqslant 1$

6. $\iint_S \sqrt{1 + x^2 + y^2}\, dS$,
S is the helicoid with vector equation $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + v\, \mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

7. $\iint_S x^2yz\, dS$,
S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

8. $\iint_S xy\, dS$,
S is the triangular region with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

9. $\iint_S yz\, dS$,
S is the part of the plane $x + y + z = 1$ that lies in the first octant

10. $\iint_S y\, dS$,
S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

11. $\iint_S x\, dS$,
S is the surface $y = x^2 + 4z$, $0 \leqslant x \leqslant 2$, $0 \leqslant z \leqslant 2$

12. $\iint_S (y^2 + z^2)\, dS$,
S is the part of the paraboloid $x = 4 - y^2 - z^2$ that lies in front of the plane $x = 0$

13. $\iint_S yz\, dS$,
S is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$

14. $\iint_S xy\, dS$,
S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

15. $\iint_S (x^2z + y^2z)\, dS$,
S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$

16. $\iint_S xyz\, dS$,
S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

17. $\iint_S (x^2y + z^2)\, dS$,
S is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$

18. $\iint_S (x^2 + y^2 + z^2)\, dS$,
S consists of the cylinder in Exercise 17 together with its top and bottom disks

19–27 ■ Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

19. $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and has upward orientation

20. $\mathbf{F}(x, y, z) = y\, \mathbf{i} + x\, \mathbf{j} + z^2\, \mathbf{k}$,
S is the helicoid of Exercise 6 with upward orientation

21. $\mathbf{F}(x, y, z) = xze^y\, \mathbf{i} - xze^y\, \mathbf{j} + z\, \mathbf{k}$,
S is the part of the plane $x + y + z = 1$ in the first octant and has downward orientation

22. $\mathbf{F}(x, y, z) = x\, \mathbf{i} + y\, \mathbf{j} + z^4\, \mathbf{k}$,
S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation

23. $\mathbf{F}(x, y, z) = x\, \mathbf{i} + y\, \mathbf{j} + z\, \mathbf{k}$,
S is the sphere $x^2 + y^2 + z^2 = 9$

24. $\mathbf{F}(x, y, z) = -y\, \mathbf{i} + x\, \mathbf{j} + 3z\, \mathbf{k}$, S is the hemisphere $z = \sqrt{16 - x^2 - y^2}$ with upward orientation

25. $\mathbf{F}(x, y, z) = y\, \mathbf{j} - z\, \mathbf{k}$,
S consists of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1$, $y = 1$

26. $\mathbf{F}(x, y, z) = x\, \mathbf{i} + y\, \mathbf{j} + 5\, \mathbf{k}$,
S is the surface of Exercise 14

27. $\mathbf{F}(x, y, z) = x\, \mathbf{i} + 2y\, \mathbf{j} + 3z\, \mathbf{k}$,
S is the cube with vertices $(\pm1, \pm1, \pm1)$

CAS **28.** Let S be the surface $z = xy$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.
(a) Evaluate $\iint_S xyz\, dS$ correct to four decimal places.
(b) Find the exact value of $\iint_S x^2yz\, dS$.

CAS **29.** Find the value of $\iint_S x^2y^2z^2\, dS$ correct to four decimal places, where S is the part of the paraboloid $z = 3 - 2x^2 - y^2$ that lies above the xy-plane.

CAS **30.** Find the flux of $\mathbf{F}(x, y, z) = \sin(xyz)\,\mathbf{i} + x^2y\,\mathbf{j} + z^2e^{x/5}\,\mathbf{k}$ across the part of the cylinder $4y^2 + z^2 = 4$ that lies above the xy-plane and between the planes $x = -2$ and $x = 2$ with upward orientation. Illustrate by using a computer algebra system to draw the cylinder and the vector field on the same screen.

31. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left.

32. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $x = k(y, z)$ and $\mathbf{n}$ is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).

33. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, if it has constant density.

34. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leqslant z \leqslant 4$, if its density function is $\rho(x, y, z) = 10 - z$.

35. (a) Give an integral expression for the moment of inertia I_z about the z-axis of a thin sheet in the shape of a surface S if the density function is ρ.
(b) Find the moment of inertia about the z-axis of the funnel in Exercise 34.

36. The conical surface $z^2 = x^2 + y^2$, $0 \leqslant z \leqslant a$, has constant density k. Find (a) the center of mass and (b) the moment of inertia about the z-axis.

37. A fluid with density 1200 flows with velocity $\mathbf{v} = y\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$. Find the rate of flow upward through the paraboloid $z = 9 - \frac{1}{4}(x^2 + y^2)$, $x^2 + y^2 \leqslant 36$.

38. A fluid has density 1500 and velocity field $\mathbf{v} = -y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$. Find the rate of flow outward through the sphere $x^2 + y^2 + z^2 = 25$.

39. Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leqslant a^2$, $z \geqslant 0$, if the electric field is $\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$.

40. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is $\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

41. The temperature at the point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \leqslant x \leqslant 4$.

42. The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

13.7 Stokes' Theorem

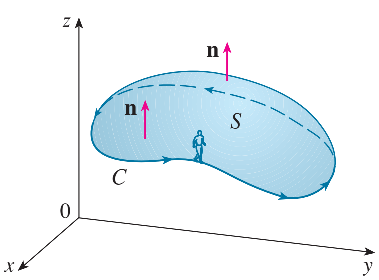


FIGURE 1

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Figure 1 shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the **positive orientation of the boundary curve C** shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.

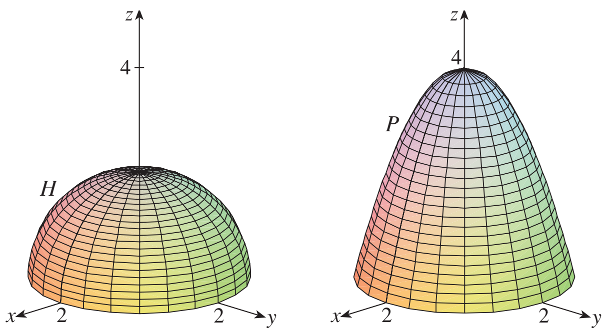
Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

13.7 Exercises

1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



2–6 ■ Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$,
S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward

3. $\mathbf{F}(x, y, z) = x^2 e^{yz}\,\mathbf{i} + y^2 e^{xz}\,\mathbf{j} + z^2 e^{xy}\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$, oriented upward

4. $\mathbf{F}(x, y, z) = (x + \tan^{-1} yz)\,\mathbf{i} + y^2 z\,\mathbf{j} + z\,\mathbf{k}$,
S is the part of the hemisphere $x = \sqrt{9 - y^2 - z^2}$ that lies inside the cylinder $y^2 + z^2 = 4$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + xy\,\mathbf{j} + x^2 yz\,\mathbf{k}$,
S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward [*Hint:* Use Equation 3.]

6. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + e^z\,\mathbf{j} + xy^2\,\mathbf{k}$,
S consists of the four sides of the pyramid with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, and $(0, 1, 0)$ that lie to the right of the xz-plane, oriented in the direction of the positive y-axis [*Hint:* Use Equation 3.]

7–10 ■ Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

7. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + (y + z^2)\,\mathbf{j} + (z + x^2)\,\mathbf{k}$,
C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

8. $\mathbf{F}(x, y, z) = e^{-x}\,\mathbf{i} + e^x\,\mathbf{j} + e^z\,\mathbf{k}$,
C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant

9. $\mathbf{F}(x, y, z) = 2z\,\mathbf{i} + 4x\,\mathbf{j} + 5y\,\mathbf{k}$,
C is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$

10. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}$,
C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant

11. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2 z\,\mathbf{i} + xy^2\,\mathbf{j} + z^2\,\mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.

(b) Graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

12. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + \frac{1}{3}x^3\,\mathbf{j} + xy\,\mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.

(b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

13–15 ■ Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

13. $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented upward

14. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + xyz\,\mathbf{k}$,
S is the part of the plane $2x + y + z = 2$ that lies in the first octant, oriented upward

15. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geqslant 0$, oriented in the direction of the positive y-axis

16. Let

$$\mathbf{F}(x, y, z) = \langle ax^3 - 3xz^2,\ x^2 y + by^3,\ cz^3 \rangle$$

Let C be the curve in Exercise 12 and consider all possible smooth surfaces S whose boundary curve is C. Find the values of a, b, and c for which $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is independent of the choice of S.

17. Calculate the work done by the force field

$$\mathbf{F}(x, y, z) = (x^x + z^2)\,\mathbf{i} + (y^y + x^2)\,\mathbf{j} + (z^z + y^2)\,\mathbf{k}$$

when a particle moves under its influence around the edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant, in a counterclockwise direction as viewed from above.

18. Evaluate $\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz$, where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \leqslant t \leqslant 2\pi$. [*Hint:* Observe that C lies on the surface $z = 2xy$.]

19. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

20. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 22 and 24 in Section 13.5 to show the following.

(a) $\int_C (f\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$

(b) $\int_C (f\nabla f) \cdot d\mathbf{r} = 0$

(c) $\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = 0$

Writing Project

Three Men and Two Theorems

▲ The photograph shows a stained-glass window at Cambridge University in honor of George Green.

Although two of the most important theorems in vector calculus are named after George Green and George Stokes, a third man, William Thomson (also known as Lord Kelvin), played a large role in the formulation, dissemination, and application of both of these results. All three men were interested in how the two theorems could help to explain and predict physical phenomena in electricity and magnetism and fluid flow. The basic facts of the story are given in the margin notes on pages 946 and 972.

Write a report on the historical origins of Green's Theorem and Stokes' Theorem. Explain the similarities and relationship between the theorems. Discuss the roles that Green, Thomson, and Stokes played in discovering these theorems and making them widely known. Show how both theorems arose from the investigation of electricity and magnetism and were later used to study a variety of physical problems.

The dictionary edited by Gillispie [2] is a good source for both biographical and scientific information. The book by Hutchinson [5] gives an account of Stokes' life and the book by Thompson [8] is a biography of Lord Kelvin. The articles by Grattan-Guinness [3] and Gray [4] and the book by Cannell [1] give background on the extraordinary life and works of Green. Additional historical and mathematical information is found in the books by Katz [6] and Kline [7].

1. D. M. Cannell, *George Green, Mathematician and Physicist 1793–1841: The Background to his Life and Work* (London: Athlone Press, 1993).

2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Green by P. J. Wallis in Volume XV and the articles on Thomson by Jed Buchwald and on Stokes by E. M. Parkinson in Volume XIII.

3. I. Grattan-Guinness, "Why did George Green write his essay of 1828 on electricity and magnetism?" *Amer. Math. Monthly*, Vol. 102 (1995), pp. 387–396.

4. J. Gray, "There was a jolly miller." *The New Scientist*, Vol. 139 (1993), pp. 24–27.

5. G. E. Hutchinson, *The Enchanted Voyage* (New Haven: Yale University Press, 1962).

6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 678–680.

7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 683–685.

8. Sylvanus P. Thompson, *The Life of Lord Kelvin* (New York: Chelsea, 1976).

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1. Thus, the net flow is outward near P_1, so div $\mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so div $\mathbf{F}(P_2) < 0$ and P_2 is a sink. We can use the formula for $\mathbf{F}$ to confirm this impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, we have div $\mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.

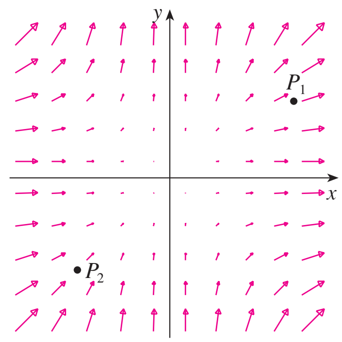
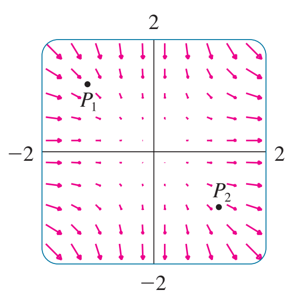


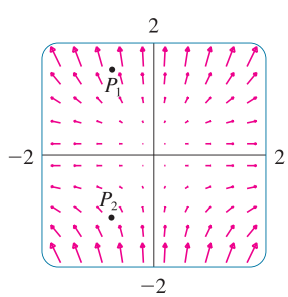
FIGURE 4
The vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

13.8 Exercises

1. A vector field $\mathbf{F}$ is shown. Use the interpretation of divergence derived in this section to determine whether div $\mathbf{F}$ is positive or negative at P_1 and at P_2.



2. (a) Are the points P_1 and P_2 sources or sinks for the vector field $\mathbf{F}$ shown in the figure? Give an explanation based solely on the picture.
(b) Given that $\mathbf{F}(x, y) = \langle x, y^2 \rangle$, use the definition of divergence to verify your answer to part (a).



3–6 ■ Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E.

3. $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$,
E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$

4. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + 3z^2\,\mathbf{k}$,
E is the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$

5. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$,
E is the solid cylinder $x^2 + y^2 \leqslant 1$, $0 \leqslant z \leqslant 1$

6. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
E is the unit ball $x^2 + y^2 + z^2 \leqslant 1$

7–15 ■ Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

7. $\mathbf{F}(x, y, z) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + yz^2\,\mathbf{k}$,
S is the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 2$

8. $\mathbf{F}(x, y, z) = x^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + xz^4\,\mathbf{k}$,
S is the surface of the box with vertices $(\pm 1, \pm 2, \pm 3)$

9. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

10. $\mathbf{F}(x, y, z) = x^3y\,\mathbf{i} - x^2y^2\,\mathbf{j} - x^2yz\,\mathbf{k}$,
S is the surface of the solid bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = -2$ and $z = 2$

11. $\mathbf{F}(x, y, z) = xy\sin z\,\mathbf{i} + \cos(xz)\,\mathbf{j} + y\cos z\,\mathbf{k}$,
S is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

12. $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k}$,
S is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane

13. $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the sphere $x^2 + y^2 + z^2 = 1$

14. $\mathbf{F}(x, y, z) = (x^3 + y\sin z)\,\mathbf{i} + (y^3 + z\sin x)\,\mathbf{j} + 3z\,\mathbf{k}$,
S is the surface of the solid bounded by the hemispheres $z = \sqrt{4 - x^2 - y^2}$, $z = \sqrt{1 - x^2 - y^2}$ and the plane $z = 0$

CAS **15.** $\mathbf{F}(x, y, z) = e^y\tan z\,\mathbf{i} + y\sqrt{3 - x^2}\,\mathbf{j} + x\sin y\,\mathbf{k}$,
S is the surface of the solid that lies above the xy-plane and below the surface $z = 2 - x^4 - y^4$, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$

CAS **16.** Use a computer algebra system to plot the vector field $\mathbf{F}(x, y, z) = \sin x\cos^2 y\,\mathbf{i} + \sin^3 y\cos^4 z\,\mathbf{j} + \sin^5 z\cos^6 x\,\mathbf{k}$ in the cube cut from the first octant by the planes $x = \pi/2$, $y = \pi/2$, and $z = \pi/2$. Then compute the flux across the surface of the cube.

17. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = z^2x\,\mathbf{i} + \left(\tfrac{1}{3}y^3 + \tan z\right)\mathbf{j} + (x^2z + y^2)\,\mathbf{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. [*Hint:* Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leqslant 1$, oriented downward, and $S_2 = S \cup S_1$.]

18. Let $\mathbf{F}(x, y, z) = z\tan^{-1}(y^2)\,\mathbf{i} + z^3\ln(x^2 + 1)\,\mathbf{j} + z\,\mathbf{k}$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

19. Verify that $\operatorname{div}\mathbf{E} = 0$ for the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\,\mathbf{x}$$

20. Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2)\,dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

21–26 ■ Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

21. $\iint_S \mathbf{a}\cdot\mathbf{n}\,dS = 0$, where $\mathbf{a}$ is a constant vector

22. $V(E) = \frac{1}{3}\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

23. $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$

24. $\iint_S D_{\mathbf{n}} f\,dS = \iiint_E \nabla^2 f\,dV$

25. $\iint_S (f\nabla g)\cdot\mathbf{n}\,dS = \iiint_E (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV$

26. $\iint_S (f\nabla g - g\nabla f)\cdot\mathbf{n}\,dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\,dV$

27. Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n}\,dS = \iiint_E \nabla f\,dV$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [*Hint:* Start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.]

28. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. We set up a coordinate system so that the xy-plane coincides with the surface of the liquid and positive values of z are measured downward into the liquid. Then the pressure at depth z is $p = \rho gz$, where g is the acceleration due to gravity (see Section 6.5). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n}\,dS$$

where $\mathbf{n}$ is the outer unit normal. Use the result of Exercise 27 to show that $\mathbf{F} = -W\mathbf{k}$, where W is the weight of the liquid displaced by the solid. (Note that $\mathbf{F}$ is directed upward because z is directed downward.) The result is *Archimedes' principle:* The buoyant force on an object equals the weight of the displaced liquid.

13.9 Summary

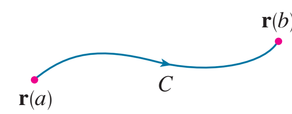
The main results of this chapter are all higher-dimensional versions of the Fundamental Theorem of Calculus. To help you remember them, we collect them together here (without hypotheses) so that you can see more easily their essential similarity. Notice that in each case we have an integral of a "derivative" over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

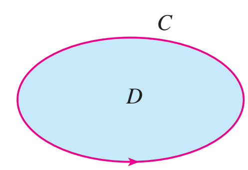
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



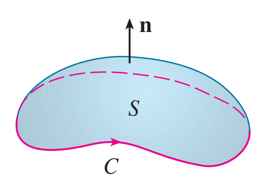
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$$



Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$